MATH 2010B Advanced Calculus I, Lecture Notes Week 7

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Last time ... ... $f(x,y)$, $f(x,y,z)$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \quad (\text{or } f_x, f_y)$$

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2} \quad (f_{xx}, f_{xy}, f_{yx}, f_{yy})$$

Mixed Derivatives Thm: $f \in C^2$ (exist and continuous)

$$\Rightarrow \quad \boxed{f_{xy} = f_{yx}}$$

One counterexample if f is not cts after 2^{nd} derivatives

Consider

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

(Ex: f is cts at $(0,0)$.)

Claim: $f_{xy}(0,0) \neq f_{yx}(0,0) \Rightarrow f \notin C^2$.

$$f_{xy}(0,0) = (f_x)_y(0,0)$$

$$:= \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \Rightarrow \quad f_x(0,k) = ?$$

For $y \neq 0$,

$$f_x(0,y) = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right)\Big|_{(0,y)}$$

$$= \frac{(x^2+y^2)\left[y(x^2-y^2) + 2x(2x)\right] - xy(x^2-y^2)\cdot(2x)}{(x^2+y^2)^2}\Bigg|_{(0,y)}$$

$$= \frac{-y^5}{y^4} = -y.$$

For $y=0$,

$$f_x(0,0) := \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

In sumary, $f_x(0,y) = -y \quad \forall y$

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-k-0}{k} = -1$$

For the right hand side,

$$f_{yx}(0,0) = (f_y)_x(0,0)$$

$$:= \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$$

For $x \neq 0$,

$$f_y(x,0) = \frac{\partial}{\partial y}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right)\Bigg|_{(x,0)}$$

$$= \frac{(x^2+y^2)[(x)(x^2-y^2)+\cdots] - \cdots}{(x^2+y^2)^2}\Bigg|_{(x,0)}$$

$$= \frac{x^5}{x^4} = x.$$

For $x=0$,

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

$\Rightarrow \quad f_y(x,0) = x, \ \forall x.$

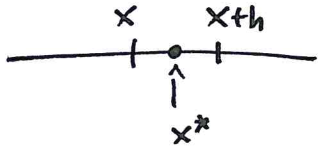
$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

Now, $f_{xy}(0,0) = -1 \neq 1 = f_{yx}(0,0)$

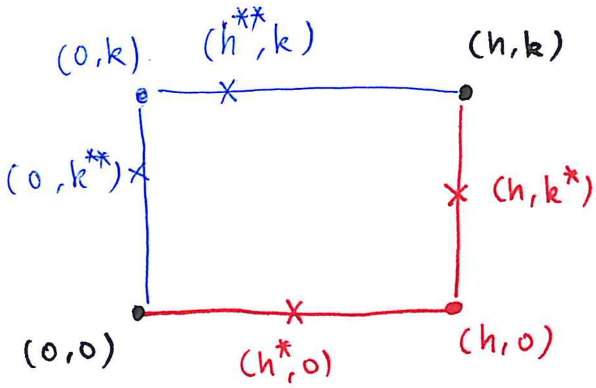
Proof of $f_{xy} = f_{yx}$ if $f \in C^2$:

Recall: Mean Value Theorem in 1-D.

$$\frac{f(x+h) - f(x)}{h} = f'(x^*)$$



Fix $(0,0)$ to be the point: ie $f_{xy}(0,0) = f_{yx}(0,0)$.



$$f(h,k) - f(0,0) = \underbrace{f(h,k) - f(h,0)} + \underbrace{f(h,0) - f(0,0)}$$

mean value thm.

$$= f_y(h,k^*)(k-0) + f_x(h^*,0)(h-0)$$

$$= \left[f_y(h,k^*)\,k + f_x(h^*,0)\,h \right]$$

$$= f(h,k) - f(0,k) + f(0,k) - f(0,0)$$

$$= \left[f_x(h^{**},k)\,h + f_y(0,k^{**})\,k \right]$$

$$\Rightarrow \quad f_x(h^{**},k)h - f_x(h^*,0)h = f_y(h,k^*)k - f_y(0,k^{**})k$$

$$\Rightarrow \quad f_{xy}(h',k')\,\cancel{hk} = f_{yx}(h'',k'')\,\cancel{hk}.$$

Since f_{xy}, f_{yx} cts $\Rightarrow$ when $h,k \to 0$, all $(h',k') \approx (h'',k'') \to (0,0)$

$$f_{xy}(0,0) = f_{yx}(0,0)$$

Eg. $f(x,y) = e^x \sin y$ $\quad$ (in C^2)

Calculate all 2^{nd} order partial derivatives.

Sol:

$$f_x = e^x \sin y \qquad f_{xx} = e^x \sin y$$

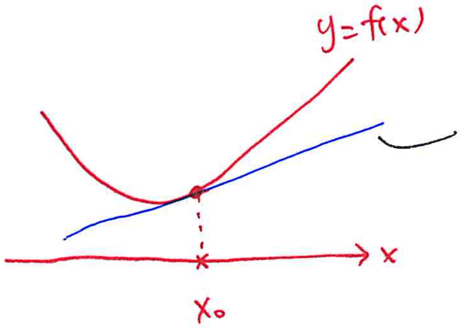
$$f_y = e^x \cos y \qquad f_{xy} = f_{yx} = e^x \cos y$$

$$f_{yy} = -e^x \sin y$$

※.

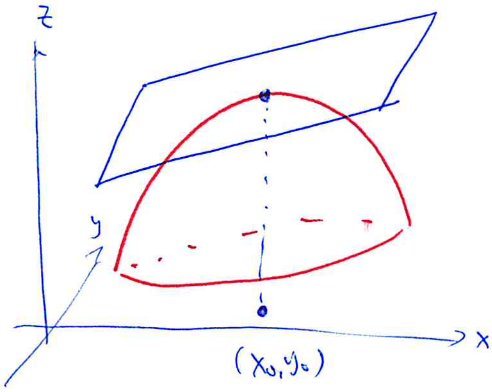
Generalizations: $f_{xyx} = f_{xxy} = f_{yxx}$ if $f \in C^3$

Recall: Tangent line to $y=f(x)$ at $x=x_0$



$$y = f(x_0) + \underbrace{f'(x_0)}_{\text{slope of tangent line.}}(x-x_0)$$

Tangent plane to $z=f(x,y)$ at (x_0,y_0)



$$z = f(x_0,y_0) + \underbrace{f_x(x_0,y_0)}(x-x_0) + \underbrace{f_y(x_0,y_0)}(y-y_0)$$

"slope" of tangent plane

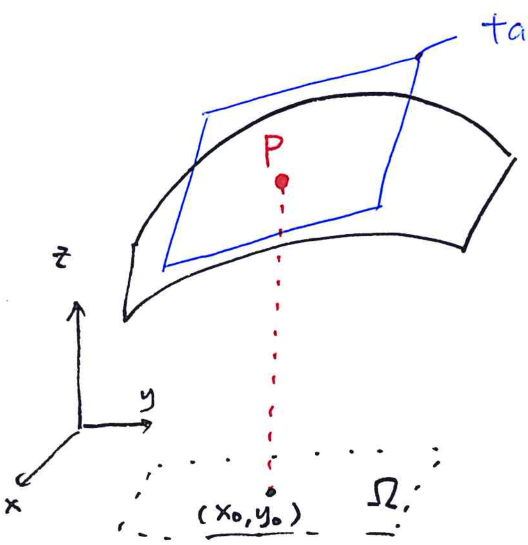
Last time... $\boxed{f_{xy} = f_{yx}}$ if $f \in C^2$

tangent plane equation.

Tangent Plane of a graphical surface $z = f(x,y)$

Setup: $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is in C^1

fix $(x_0, y_0) \in \Omega$.



Q: How to the equation of this tangent plane?

A:
(*)
$$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

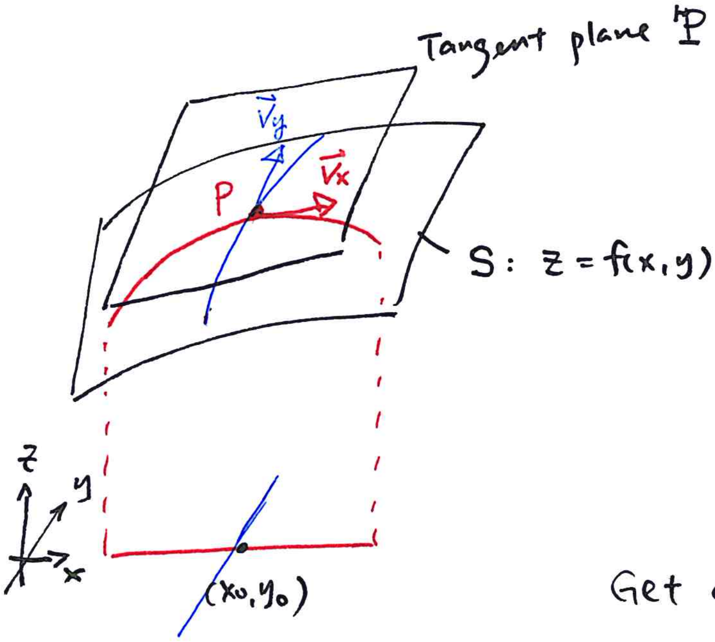
The plane (*) agrees with the $z = f(x,y)$ up to 1^{st} order at (x_0, y_0).

Plane: $z = g(x,y)$

Surface: $z = f(x,y)$

then
$$\begin{cases} g(x_0,y_0) = f(x_0,y_0) \\ g_x(x_0,y_0) = f_x(x_0,y_0) \\ g_y(x_0,y_0) = f_y(x_0,y_0) \end{cases}$$

Q: How to derive (*)?



$\mathbb{P}$ contains P and parallel to $\vec{v}_x$, and $\vec{v}_y$.

Now: $P = (x_0, y_0, f(x_0,y_0))$

red curve: $\gamma_1(t) = (x_0+t, y_0, f(x_0+t, y_0))$

$\gamma_1(0) = P$; $\gamma_1'(0) = \vec{v}_x = (1, 0, f_x(x_0,y_0))$

blue curve: $\gamma_2(t) = (x_0, y_0+t, f(x_0, y_0+t))$

$\gamma_2(0) = P$; $\gamma_2'(0) = \vec{v}_y = (0, 1, f_y(x_0,y_0))$

Get a normal vector at P,

$$\vec{n} = \vec{v}_x \times \vec{v}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1)$$

Tangent plane equation:

$$(\vec{x} - P)\cdot\vec{n} = 0$$

i.e. $$(x-x_0, y-y_0, z-f(x_0,y_0))\cdot(-f_x(x_0,y_0), -f_y(x_0,y_0), 1) = 0$$

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + z - f(x_0,y_0) = 0$$

(*) $$\boxed{z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

tangent plane at P.

Example: Find the equation of the tangent plane for

$$z = x^2+y^2 \quad \text{at} \quad (1,0,1).$$

Solution: Check: ① $z = x^2+y^2 =: f(x,y)$ graphical surface of f

② $P = (1,0,1)$ lies on the surface

(ie $1 = 1^2+0^2$ ✓)

Now, $(x_0,y_0) = (1,0)$ and we know

Eqn: $$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\left.\begin{aligned} f(x,y) &= x^2+y^2 \\ f_x &= 2x \\ f_y &= 2y \end{aligned}\right\} \Rightarrow \begin{aligned} f(1,0) &= 1 \\ f_x(1,0) &= 2 \\ f_y(1,0) &= 0. \end{aligned}$$

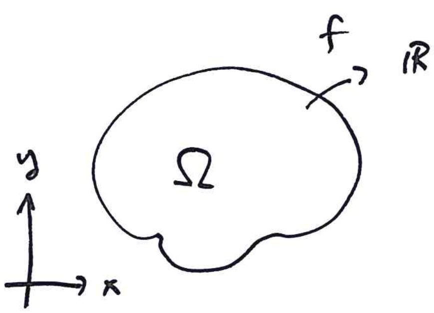
$$\Rightarrow \quad z = 1 + 2\cdot(x-1) + 0\cdot(y-0)$$

i.e. $$z = 1 + 2(x-1)$$

ie $$\boxed{2x - z = 1}$$

Applications (Min / Max Problems)

Q: Given $f: \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, how do we find the "maximum" & "minimum" (extremum) of f?

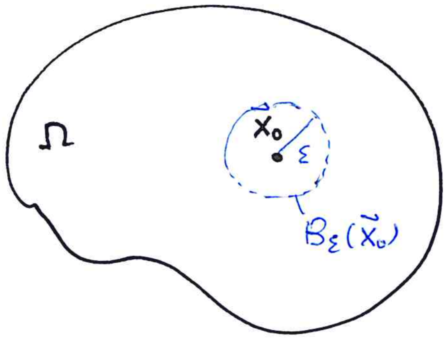


$$\boxed{\max_{\Omega}/\min_{\Omega} f(x,y)}$$

Def$^{\underline{n}}$: A point $\vec{x}_0 \in \Omega$ is said to be

- a local maximum of f if $f(\vec{x}_0) \geq f(\vec{x})$

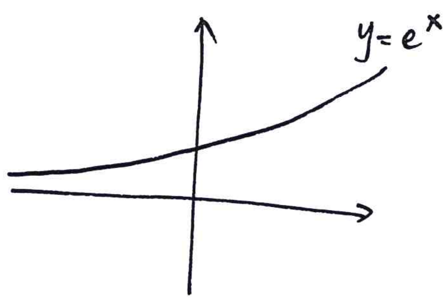
$$\forall\ \vec{x} \in \Omega \cap B_\varepsilon(\vec{x}_0) \text{ for some } \varepsilon > 0$$



- a global maximum of f if $f(\vec{x}_0) \geq f(\vec{x})\ \forall \vec{x} \in \Omega$

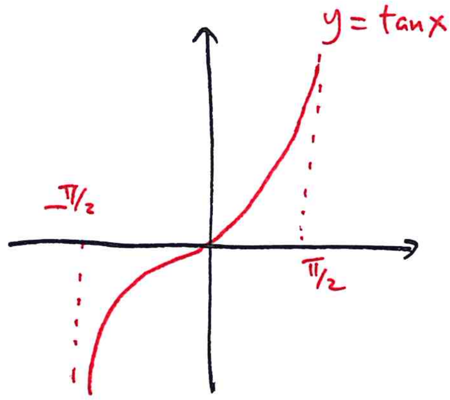
Note: In some cases, min/max f do not exist.

(1D): bad examples.



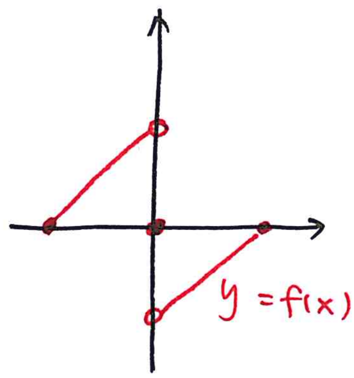
no min/max

interval is infinite



no min/max.

$(-\frac{\pi}{2}, \frac{\pi}{2})$ is open



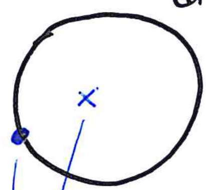
no min/max.

f is not cts

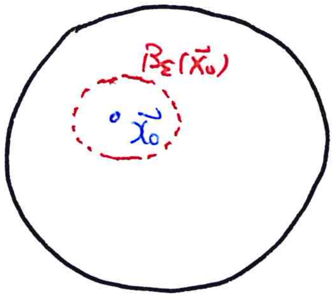
Extreme Value Theorem:

If $f: \Omega \to \mathbb{R}$ is a ① continuous function on a ② bounded and ③ closed $\Omega \subseteq \mathbb{R}^n$,

then the global min/max of f exist.



First Derivative Test $\quad f = f(x,y)$



If (1) $\vec{x}_0 \in \Omega$ is an interior point

(i.e. $\exists\, \varepsilon > 0$, s.t $B_\varepsilon(\vec{x}_0) \subset \Omega$)

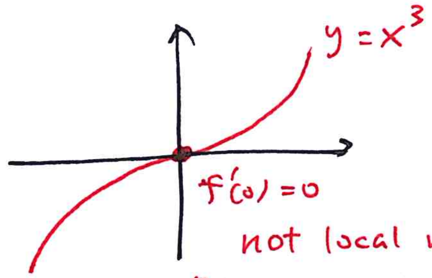
(2) $\vec{x}_0$ is a local min/max for f

where $f : \Omega \to \mathbb{R}$ is in C^1.

then $\cancel{f_x(x_0, y_0}$ $\boxed{f_x(\vec{x}_0) = 0 = f_y(\vec{x}_0)}$

all the 1^{st} order partial derivatives vanish at $\vec{x}_0$.

Note: $\Leftarrow$ is false!   1D - example: $y = f(x) = x^3$, $x_0 = 0$



not local max/min

"inflection pt"

Def$^{\underline{n}}$: An interior pt $\vec{x}_0 \in \Omega$ is a critical point of f

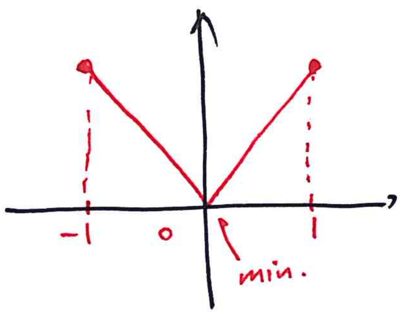
if either (i) f does not have first partial derivative at $\vec{x}_0$

(ie. either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist)

or (ii) $\boxed{f_x(x_0, y_0) = 0 = f_y(x_0, y_0)}$.

e.g. (why need (i))

1D - example: $y = f(x) = |x|$.



0 is a min but f is not differentiable at 0.

Observe: Goal: $\min/\max_{\Omega} f(x,y)$

the min/max $\vec{x}_0 \in \Omega$ — $\vec{x}_0 \in \partial\Omega$

— $\vec{x}_0 \notin \partial\Omega$ and is a critical pt.

Example A: Find the minimum of $f(x,y) = x^2+y^2-4y+9$ on $\Omega = \mathbb{R}^2$

$$\min_{\mathbb{R}^2} f(x,y)$$

Observations: (1) f is C^1 on $\mathbb{R}^2$

(2) $\partial\Omega = \emptyset$ $\Rightarrow$ every pt is interior.

$\Rightarrow$ The min. (if it exists) must be a critical point

Q: Where are the critical points?

ie $(x_0,y_0)\in\Omega$ s.t. $\begin{cases} f_x(x_0,y_0)=0 \\ f_y(x_0,y_0)=0 \end{cases}$

$$f = x^2+y^2-4y+9$$

$$\begin{cases} f_x = 2x = 0 \\ f_y = 2y-4 = 0 \end{cases} \Rightarrow \begin{matrix} x=0 \\ y=2 \end{matrix}$$ is the only solution

$\Rightarrow$ 1 critical point $(x_0,y_0)=(0,2)$

Q: Is $(x_0,y_0)=(0,2)$ a local min / max?

$f = x^2+y^2-4y+9$, $(x,y)\to\infty \Rightarrow f \to +\infty$

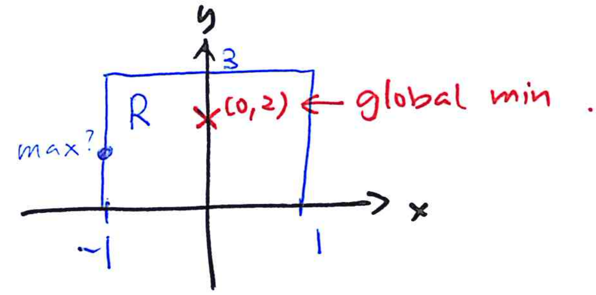
$x^2+y^2\to\infty$

$\Rightarrow$ $(0,2)$ is a ~~minima~~ minimum

with $f(0,2) = 4-8+9 = 5$

[See: $f = x^2+(y-2)^2+5 \geq 5$]

Example B : $\boxed{\max_R / \min_R f}$ where R is a rectangle (closed)



Note: R closed + bdd $\Rightarrow$ max/min f exist.

$\min_R f = 5$ at $(0,2)$

since it is a global minimum on $\mathbb{R}^2 \supset R$

Q: What about the maximum?

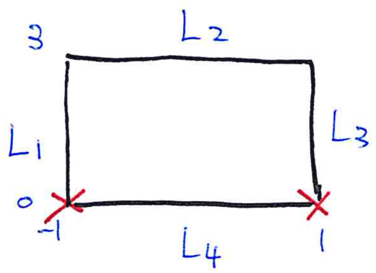
- only 1 critical pt. $(0,2)$ in the interior, which is a min.

$\Rightarrow$ $\vec{x}_0$ max pt $\in \partial R$

$\Rightarrow$ reduce the problem to

$$\max_{\partial R} f \qquad \text{1D-problem!!}$$

∂R



On L_1: $f(-1,y) = y^2 - 4y + 10$

$$\max_{y\in[0,3]} (y^2-4y+10) \overset{\text{Ex:}}{=} 10.$$

On L_2: $f(x,3) = x^2 + 6$

$$\max_{x\in[-1,1]} (x^2+6) \overset{\text{Ex:}}{=} 7$$

On L_3: $f(1,y) = y^2 - 4y + 10$

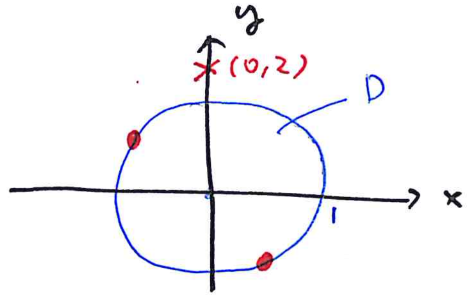
Same as L_1 max $= 10$

On L_4: $f(x,0) = x^2 + 9$

$$\max_{x\in[-1,1]} (x^2+9) = 10$$

max $= 10$ achieved at $(-1,0)$ and $(1,0)$.

Example C : $\max_D/\min_D f$ where D = closed unit disk



Sol: There is no critical pts in the interior of D.

So, min/max $\vec{x}_0 \in \partial D$.

Use polar coordinates : $\partial D = \{ (r,\theta) \mid r = 1 \}$

$$f = x^2 + y^2 - 4y + 9$$

$$= (r\cos\theta)^2 + (r\sin\theta)^2 - 4(r\sin\theta) + 9$$

$$f(r,\theta) = r^2 - 4r\sin\theta + 9 \qquad \text{(in polar coord.)}$$

On ∂D, $r = 1$,

$$f(1,\theta) = 1 - 4\sin\theta + 9 = 10 - 4\sin\theta .$$

$$\boxed{\min_\theta/\max_\theta \ (10 - 4\sin\theta)} \quad \underline{\text{vs}} \quad \min/\max_{x^2+y^2=1} \ x^2+y^2-4y+9$$

since $|\sin\theta| \le 1$

min happens when $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

max happens when $\sin\theta = -1$, $\theta = \frac{3\pi}{2}$.

$$\boxed{\begin{aligned} \min_D f &= 6 \quad \text{at } (0,1) \\ \max_D f &= 14 \quad \text{at } (0,-1). \end{aligned}}$$

Q: